Mathematical Logicism

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This talk is based on a chapter of a book in progress, tentatively entitled *Minimal Foundations: A Leibnizian Theory of Metaphysical Analysis*. In the book I develop a theory of metaphysical analysis centered around two logically related principles. One, *metaphysical foundationalism*, says that every entity has a unique metaphysical analysis in terms of the fundamental entities. The focus of this talk will be a different idea, that turns out to be equivalent to the uniqueness part in the former principle. It says:

Global Logicism

All necessities amount to logical truths after metaphysical analysis.

By a necessity I mean a proposition that is necessary in the broadest sense. Notions like logical truth and analysis (and logical form and definition) are notions that I am applying to non-linguistic entities like propositions, properties, individuals and so on. Since these notions are usually applied to entities that have at least as much structure as expressions in a language have they need a more careful treatment, which is provided in the book.

1. Global Logicism

One can find canonical statements of Global Logicism in Leibniz:

> Necessary truths are those that can be demonstrated through an analysis of terms, so that in the end they become identities [...] (from *Contingent Truths*)

According to Leibniz, every proposition can be given an analysis in terms of the fundamental (simple) entities: it is a logical function of them. If the proposition is a necessary one, then this logical function has the form of a logical truth.

The role of metaphysical analysis is important here. Take, for instance, the necessary truth that *a whole is bigger than its part.* On its surface, it is not have the form of a logical truth. But, says Leibniz, once all definitions (identities) are applied so that it is expressed as a function of the fundamental entities, it *becomes* a logical truth. In *First Truths* he writes:

> all other truths are reducible to first ones through definitions, that is, by resolving notions into their simpler components [...]
>
> I choose as an example this:
>
> > A whole is bigger than its part, or
> >
> > A part is smaller than the whole.
>
> This is easily demonstrated from the definition of 'smaller' or 'bigger' together with the basic axiom, that is, the axiom of identity. Here is a definition of 'smaller than':
>
> > For x to be smaller than y is for x to be equal to a part of y (which is bigger).

There are various challenges to global logicism.

- Color: it seems to be necessary that nothing can be red all over and blue all over, but it is not obvious how this reduces to a logical truth.
- Essence: it is hard to see how it could be a logical truth that Saul Kripke has the parents he in fact has, or that a particular table is constituted substantially out of some particular hunk of wood.
- Ethics: There are necessary truths in ethics which appear hard to assimilate to logic.
- Modality: truths about what is necessary and possible are often themselves necessary or possible, but can these always be reduced to logic?

2. Challenges from mathematics

The most serious opposition to global logicism came from Kant, and his insistence that the truths of arithmetic are necessary but not analytic. Bolzano followed Kant in this belief about mathematics but made these ideas precise, however, by giving an explicit definition of what it means for a *proposition* (as opposed to a sentence of a language) to be analytic: that, once it is analyzed into its metaphysically simple constituents, the proposition is a *formal* logical truth in the sense that it is true under all substitutions of the simple constituents of the proposition for entities of the same type.[1] However Bolzano's argument rested on the assumption that once metaphysically analyzed a claim like $2 + 3 = 5$ has 2, 3 and 5 as simple constituents; if he were right then it would be a formal logical truth only if $\forall xyz(x + y = z)$, which is evidently false.

In the talk I want to defend mathematical logicism:

Mathematical Logicism

All truths of mathematics reduce to logical truths after metaphysical analysis.

Actually this is stronger than is needed for a defence of global logicism: all that is really needed is that all *necessities* of mathematics reduce to logic. As some people understand the word 'logic', certain mathematical statements, such as the continuum hypothesis, cannot be truths of logic in virtue of their truth values being (seemingly) indeterminate, and not knowable *a priori*. Someone who wishes to explore Global Logicism using this stronger sense of logic need only defend the weaker claim. If the continuum hypotheses, appropriately understood as a certain mathematical proposition, is genuinely indeterminate then it is not necessary in the broadest sense, and so it is perfectly compatible with global logicism if it does not count as logically true or logically false.

3. Platonic facts and patterns in nature

As arithmetic is studied today, it is typically formalized in a first-order language that have names, like 0, standing for abstract 'platonic' numbers, quantifiers that range over these platonic objects, and predicates standing for arithmetical relations between the platonic objects corresponding to comparisons, successor, addition and multiplication. If we imagine that numbers are fundamental, and the aforementioned arithmetical relations are fundamental, then a claim like *7 is prime* can be metaphysically analyzed as follows:

[1] It is a 'formal' truth, because all propositions with the same logical form are true.

> For any platonic numbers x and y, if x times y is the seventh successor of 0, then either x or y is the successor of 0.

This sentence is not a logical truth. If there were fundamental platonic individuals, with these sorts of relations between them, Global Logicism should say that it is possible that the number 7 is not prime. Indeed, if there really were fundamental platonic numbers, I see no reason why it should not be possible, in the broadest logical sense, for the number 7 to have led a completely different life as a Roman emperor, and so not be prime.

But prior to these modern platonistic formalizations disciplines like arithmetic, geometry and analysis were more closely related to patterns in nature to do with number, shape and quantity. A statement like

> If you have two goats and three sheep, and no goat is a sheep, then you have five things that are sheep or goats.

is an instance of a much more general pattern in nature, namely:

> For any F and G, if you have two Fs and three G, and no F is G, then you have five things that are F or G.

This pattern is arguably the "real" mathematical fact we should care about, and the platonic fact about abstract numbers, $2 + 3 = 5$, is at best a representation of it. If the platonic numbers 2, 3 or 5 had been Roman emperors then the platonic fact would be no more, yet the pattern would still obtain.

It is also true that it is the general patterns that are directly related to the applications to the world. The particular claim about goats and sheep is an instance of the general pattern claim, and so follows directly by logic, whereas the platonic fact can only imply this particular claim with the help of metaphysically substantive bridge principles, connecting the abstract and the physical realms.

So I claim:

> The theoretically central claims of mathematics are the general patterns in nature, not the platonic truths.
>
> The true general mathematical patterns, suitably analyzed, are just truths of logic.

As for platonic facts: I am happy to say that there are no platonic numbers, for they are theoretically otiose. So there are no platonic necessities to be explained. If you insist on platonic objects, however, they could be as modally flexible as Roman emperors with respect to the broadest necessity.[2]

4. Mathematical Logicism

I propose to identify mathematical truths with truths about such patterns, expressed using higher-order resources rather than a special ontology of abstract objects.

[2]This is not to completely rule out some non-trivial necessities involving platonic objects. A naive mereologist might want to say that Julius Caesar is metaphysically defined from his mereologically atomic parts, and so there may be logical connections between Caesar and his atomic parts after all (this relates to the idea that parthood is a logical notion). If one took the successor operation to be logical and 0 to be fundamental, 7 could be metaphysically analyzed in terms of 0, and so there would be logical necessities involving 7 after metaphysical analysis.

Core Idea

Mathematical facts are general patterns in nature, and can be encoded as truths about *maximally specific consistent structural properties.*

A structural property is, very roughly, a higher-order property of relations that characterizes them entirely by the structural properties of their extensions. Examples:

- $\lambda R.R$ is an equivalence relation.
- $\lambda R.R$ is a well-order
- $\lambda R.R$ is a well-order of order type ω.

Of these only the last one is *maximally specific*: it either entails any other structural property or entails its negation. Given a suitably strong theory of logical possibility, it is also consistent: it is possible that there is an ω structure.

Some non-examples:

- $\lambda R.R$ is a well-order of order type ω whose first element is the platonic number 0.
- $\lambda R.R$ is has order-type ω if Trump is president in 2026, and order-type ω_1 if he does not.

Structural properties should be invariant when we replace the individuals in a structure in a way that preserves the structure of a binary relation. The first property lacks this feature. The second necessarily has this feature, but structural properties should be also invariant if we permute the worlds. Structural properties, then, can be defined as those that are invariant under all permutations of individuals and worlds, understood as logical relations.

Definition of structure

By a structure of a given signature (i.e. list of arities) I will mean a property of individuals, D, and first-order relations $R_1, \dots R_n$ of the specified arities whose fields are all contained in the extension of D.

A higher-order relation Q, taking a list of properties and relations of the given signature as arguments, is structural when ρQQ holds for every logical relation based on any pair of a permutation $\rho^e : e \to e \to t$ and complete Boolean automorphism $\rho^t : t \to t \to t$

A structure is a structural property that is possibly instantiated, and entails any other structural property or its negation.

Structural properties, I claim, are always "pure", and are the sorts of things that can be used in metaphysically analysing one thing in terms of some others. But not all pure properties need be structural. A permutation of worlds when applied to propositions always preserves conjunctions of propositions. But, if the Barcan formula fails, universals are not identical to the conjunctions of their instances, and so these permutations need not preserve the universal quantifier.

To recover mathematics, logicists have typically needed to make special extra assumptions of mathematical plenitude: the axiom of infinity, that there is an abundance of sets, or categories or what have you. My approach is compatible with finitism about individuals, the necessary principles of plenitude of maximally specific structural properties falls out of the existing theory of metaphysical analysis.

So, for example, one can associate to an arithmetical statement quantifying over numbers a general pattern expressed by quantifying into the position of numerical quantifiers. Here is the road map.

- **Arithmetic**: count numerical quantifiers (e.g. "λF.there are N Fs"). The maximally specific structural properties of plain collections of objects are just their number.
- **Analysis**: mass numerical quantifiers (e.g. "λFG. there is α times as much F as G"). These are maximally specific structural properties of collections endowed with parthood and size.
- **Ordinals**: maximally specific structural properties of well-orders, that is, properties of having a certain order-type.
- **Sets**: maximally specific structural properties of well-founded extensional trees.

What matters is that the relevant pattern can obtain whether or not there are special abstract objects corresponding to it. If seven pebbles cannot be arranged into a non-trivial rectangle, that pattern is there in nature independently of whether the number 7 is taken to be an abstract object.

For illustration, here are the definitions of *set formation*, *membership* and *set* in this framework. Let G be a rigid property of set-like structures. Then $\{G\}$ is the property a tree u has just in case every G-set is instantiated, and the immediate subtrees of u are exactly the trees instantiating some member of G.

Set Formation.

$$\{G\} := \lambda u\Big(\mathrm{Tree}(u) \wedge \forall X\,(GX \to \exists x\, Xx) \wedge \forall x\big(x \lhd u \leftrightarrow \exists X\,(GX \wedge Xx)\big)\Big).$$

here $\lhd$ is the relation of being an immediate subtree.

From this I adopt Frege's definition of membership:

Membership.

$$\in := \lambda XY.\exists^{Rg} G\big(GX \wedge Y = \{G\}\big).$$

Definition of Set

An entity X is a *set* iff, for every property V, whenever V is closed under consistent set formation—that is, whenever for every rigid consistent property G, if every instance of G is a V-entity, then the entity $\{x \mid Gx\}$ is also a V-entity—it follows that X is a V-entity.

Formalization.

$$\mathrm{Set} := \lambda X.\forall V\Big(\Big[\forall G\Big(\big(\mathrm{Rigid}(G) \wedge \mathrm{Consistent}(G) \wedge \forall y(Gy \to Vy)\big) \to V(\{x \mid Gx\})\Big)\Big] \to VX\Big).$$

"consistent" is a nod to Cantor's notion of a consistent multiplicity, namely a collection of sets that could exist all together at once. In the present formalism, it can be defined as a rigid collection of maximally specific descriptions of set structures, that could have all be instantiated simultaneously.

The important point is not merely that these definitions can be given in purely logical language, but that the required modal plenitude falls out of the broader foundational picture: Humean recombination, together with the claim that structural properties are logical and therefore available "for free" in metaphysical analysis, supplies the modal room needed for the logicist constructions.